On head-wave amplitudes

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The head-wave contribution to a reflection is investigated by two different methods and it is shown that the new result presented by Lerche and Hill [J. Math. Phys. **26**, 1420 (1985)] for the head-wave amplitude is in error due to the use of an inappropriate mathematical method.

I. INTRODUCTION

In a recent paper, Lerche and Hill[1] have investigated the amplitudes of reflections and head waves from a rough interface. As a prerequisite, they considered a smooth interface and derived an asymptotic result for the amplitude of the head-wave contribution (LH30—henceforth we use the abbreviation LH to refer to Lerche and Hill and equations therein). They comment that their result differs from the expression in Aki and Richards[2] [Eq. (6.25)] by a factor $(2/e)^{1/2}$ and that the evaluation of Aki "involves further approximations whereby the factor $(2/e)^{1/2}$ is lost." This result and statement are surprising as the asymptotic results for the head-wave amplitude derived by a variety of distinct techniques, e.g., asymptotic ray theory,[3] Cagniard–de Hoop–Pekeris method[4–8] as well as the usual branch cut evaluations by asymptotic techniques,[2,9,10] agree. Incidentally, we find that expression (LH30) is in error by a factor of $(2e)^{-1/2}$ [not $(2/e)^{1/2}$], as the result in Aki and Richards[2] also contains an error of a factor of 2. In this paper, we rederive the head-wave amplitude by two techniques (to establish beyond doubt the correct result) and identify the source of error in LH.

II. BRANCH-CUT EVALUATION

The expression for the pressure field of the reflected wave is given by

$$P_{\text{refl}} = i\int_0^\infty B\,\frac{J_0(KR)e^{ik_1(z+z_0)}}{k_1}\,K\,dK, \qquad \text{(LH6)}$$

where the variables are defined in LH and for brevity we have not substituted for the reflection coefficient, B (LH5). This integral can be decomposed and the head-wave contribution comes from the term

$$I_2 \simeq e^{-i(\pi/4)}\left(\frac{\omega s_1}{2\pi R}\right)^{1/2}\int_0^{\pi_2}\frac{d\theta\cos\theta(\sin\theta)^{1/2}}{a+b\sin^2\theta}\times(\sin^2\theta_c-\sin^2\theta)^{1/2}e^{iN\cos(\theta-\phi)} \qquad \text{(LH14)}$$

$$= e^{-i(\pi/4)}\left(\frac{\omega s_1}{2\pi R}\right)^{1/2}\int_0^{\pi_2}\frac{d\theta\cos\theta(\sin\theta)^{1/2}}{a+b\sin^2\theta}\times\exp[iN\cos(\theta-\phi)+\tfrac{1}{2}\ln(\sin^2\theta_c-\sin^2\theta)], \qquad \text{(LH18)}$$

where, using LH's notation, we have made some trivial substitutions to simplify. LH evaluated the second expression (LH18) by finding the stationary points of the exponent. The traditional method of evaluating the branch cut integral in expression (LH14) is to change the variable of integration to the branch cut radical,[9,10] i.e.,

$$\nu = (\sin^2\theta_c - \sin^2\theta)^{1/2} \qquad (1)$$

and convert it to a saddle-point integral. With this substitution (1), the exponent in (LH14) becomes

$$\chi = iN[\cos\phi(1-\sin^2\theta_c+\nu^2)^{1/2} + \sin\phi(\sin^2\theta_c-\nu^2)^{1/2}], \qquad (2)$$

which has a stationary point, i.e., $d\chi/d\nu = 0$ at $\nu = 0$. At this saddle point we have

$$\chi = iN\cos(\phi-\theta_c)$$

and

$$\frac{d^2\chi}{d\nu^2} = -\frac{iN\sin(\phi-\theta_c)}{\sin\theta_c\cos\theta_c}. \qquad (3)$$

The integral can be evaluated using the principle of stationary phase without distorting the path of integration which runs from $\nu = \sin\theta_c$ to $\nu = 0$ and then to $\nu = i\cos\theta_c$. Using the second-order saddle-point method, we obtain the asymptotic result

$$I_2 \simeq -\frac{1}{\omega s_1}\cdot\frac{\sin\theta_c}{R^{1/2}}\left(\frac{\cos\theta_c}{\mu\sin(\phi-\theta_c)}\right)^{3/2}\times\frac{\exp[iN\cos(\theta_c-\phi)]}{a+b\sin^2\theta_c}. \qquad (4)$$

This expression should be compared with (LH27) and is seen to differ by a factor $-(2e)^{1/2}$ [the negative sign appears to be a trivial error corrected in (LH30)]. The expression for the head-wave contribution is then

$$P_{\text{head}} = \frac{i}{\omega}\cdot\frac{2\rho_1}{\rho_2}\cdot\frac{s_2}{s_1^2-s_2^2}\cdot\frac{\exp(i\omega\tau_h)}{R^{1/2}L^{3/2}} \qquad (5)$$

compared with (LH30). This expression (5) agrees with Aki and Richards[2] [Eq. (6.25)] apart from a factor of 2 that has been lost between Eqs. (6.23) and (6.24) of that book. Next we discuss why LH's evaluation is in error and then confirm expression (5) by another method.

III. SADDLE-POINT EVALUATION

To evaluate expression (LH18), LH first determine the stationary points of the exponent. The significant one exists at

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$$\theta_2 = \theta_c + i/2N \sin(\phi - \theta_c) + O(N^{-2}) \qquad \text{(LH22)}$$

and the second derivative is

$$\frac{d^2\psi}{d\theta^2} = 2N^2 \sin^2(\theta_c - \phi) + O(N). \qquad \text{(LH26)}$$

The contour of integration is distorted to pass through the saddle point θ_2 and the integral evaluated by the saddle-point method. Unfortunately, although θ_2 is a stationary-phase point, the saddle-point method is inappropriate. A basic requirement for the saddle-point method is that the Taylor expansion of the exponent at the stationary-phase point should be a good approximation within the width of the saddle point where the main contribution to the integral arises. From the second derivative of the exponent (LH26), we see that the width of saddle is of order $1/|N \sin(\theta_c - \phi)|$. Higher derivatives of the exponent are given by

$$\frac{d^n\psi}{d\theta^n} = -\frac{1}{2}(n-1)!(2iN)^n \sin^n(\phi - \theta_c)[1 + O(N^{-1})]. \qquad (6)$$

It is readily apparent that within the width of the saddle point, all terms in a Taylor expansion of the exponent are of similar magnitude and importance. Notice that the distance between the saddle point θ_2 (LH22) and the branch point θ_c is comparable with the saddle width, and at the branch point the exponent is singular. The Taylor expansion is therefore not valid over the required range. The saddle-point method cannot be used to evaluate the integral and LH's result (LH27) is in error. Rather than leading to a "small" correction by $(2/e)^{1/2}$, their technique is inappropriate and leads to a significant error of $(2e)^{-1/2}$.

LH have evaluated the integral I_2 by a technique equivalent to the Stirling approximation for the gamma function. Then we have

$$\Gamma(n+1) = \int_0^\infty x^n e^{-x}\, dx \qquad \text{(7a)}$$

$$\simeq n^n e^{-n} \int_{-\infty}^{\infty} e^{-(1/2n)(x-n)^2}\, dx \qquad \text{(7b)}$$

$$= (2n\pi)^{1/2} n^n e^{-n}. \qquad \text{(7c)}$$

Expression (7a) is the integral definition of the gamma function which is evaluated by the second-order saddle point approximation (7b) to give the Stirling approximation (7c). This approximation is valid for $n \gg 1$ when the saddle point at $x = n$ is well away from the origin. But for $n = \frac{1}{2}$, we know $\Gamma(\frac{3}{2}) = \pi^{1/2}/2$ exactly, whereas the Stirling approximation (7c) gives $(\pi/2e)^{1/2}$, i.e., a factor of $(2/e)^{1/2}$. The other factor of 2 comes from approximating the integral on both sides of the branch point by the semi-infinite integral [as in (7a)].

IV. THE CAGNIARD–DE HOOP–PEKERIS METHOD

An alternative technique for investigating the head-wave contribution is the Cagniard–de Hoop–Pekeris method.[4–8] Rather than evaluate (LH6) asymptotically, we take the inverse Fourier transform with respect to frequency and obtain the impulse response. The head-wave contribution can then be investigated by a first-motion approximation, i.e., a Taylor expansion about the head-wave discontinuity, and is found to be the inverse Fourier transform of (5).

We outline very briefly the Cagniard technique. Using the symmetries of the Bessel function, expression (LH6) can be rewritten (ω real)

$$P_{\text{refl}} = \frac{i|\omega|}{2} \int_{-\infty}^{\infty} B \frac{H_0^{(1)}(\omega p R) e^{i\omega q_1 (z+z_0)}}{q_1} p\, dp, \qquad (8)$$

where $\omega q_1 = k_1$ and $\omega p = K$. The p-contour is distorted so that $\text{Im}(pR + q_1(z+z_0)) = 0$, the so-called Cagniard contour. We use

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} H_0^{(1)}(a\omega) e^{-i\omega t}\, d\omega = \begin{cases} -\dfrac{2i}{\pi(t^2 - a^2)^{1/2}}, & \text{for } |t| > a, \\ 0, & \text{for } |t| < a, \end{cases} \qquad (9)$$

to take the inverse Fourier transform of (8) and obtain

$$P_{\text{refl}} = \frac{2}{\pi} \cdot \frac{d}{dt} \text{Im} \int_{c_-} \frac{Bp}{q_1} \frac{H(t - pR - q_1(z+z_0))}{[(t - q_1(z+z_0))^2 - p^2R^2]^{1/2}} dp, \qquad (10)$$

where by symmetry we have restricted the Cagniard contour to the fourth quadrant. This expression is zero for $t < \tau_h$ (as all terms in the integrand are real) and nonzero for $t > \tau_h$ as B becomes complex at the branch point, $p = s_2$. Expanding about $p = s_2$ for $t \gtrsim \tau_h$, the important terms in the integrand are for the factor $(p - s_2)^{1/2}$ from the branch cut in B, and $(t - \tau_h - L(p - s_2))^{-1/2}$ from the denominator. Treating the other terms as constant and evaluating the integral, we obtain

$$P_{\text{head}} \simeq -\frac{2\rho_1}{\rho_2} \cdot \frac{s_2}{s_1^2 - s_2^2} \cdot \frac{1}{R^{1/2} L^{3/2}} H(t - \tau_h), \qquad (11)$$

which is consistent with the inverse Fourier transform of (5). Including higher-order terms in the Taylor expansion of the integrand (10) results in contributions to the head wave with higher-order discontinuities, e.g., $(t - \tau_h) \times H(t - \tau_h)$, but this does not modify the asymptotic result (5).

V. CONCLUSIONS

It has been shown that the new result given by Lerche and Hill[1] for the head-wave contribution to a reflection is incorrect by a factor $(2e)^{-1/2}$. The correct result is confirmed using two different but standard techniques. The error occurred in Lerche and Hill due to the inappropriate use of the saddle-point method. Their method is equivalent to using the Stirling approximation for $\Gamma(n+1)$ with $n = \frac{1}{2}$ [introducing an error of $(2/e)^{1/2}$] and reducing an integral on both sides of the branch point to one side (causing a further error of $\frac{1}{2}$). Fortunately the error in Sec. II of LH does not affect the rest of the paper. The results in Sec. III are still valid.[11]

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